Linear theory of Faraday instability in viscous fluids

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1 Introduction

When a column of liquid is subjected to vertical oscillations and surface is asymmetrically perturbed, then some Fourier components of the perturbation become unstable. This phenomenon is known since the observations of Faraday. Here we analyse the same hydrodynamic problem in the case of viscous fluids under linearized approximations. We address this problems in the following stages:

- Derive linearized fluid dynamic equations for the perturbation in displacement and vertical velocity field on the surface and transform those equations for Fourier components in space-domain
- In response to the externally induced oscillatory homogeneous force, each Fourier component(corresponding to a unique wave-vector) of space-domain is analysed using Floquet theory in time-domain and a recursive equation for the components of Floquet spectra of surface's displacement field with any wave-vector is obtained
- Representing the solution to the recursive equation obtained by an equivalent eigenvalue problem for an infinite dimensional matrix.
- Truncating the matrix according to certain approximation schemes and certain constraints, then computing the eigenvalues numerically leading to instability boundaries.

From the results of the last stage, we obtain a remarkable physical result that in case of viscous fluids, the surface stays stable and flat until the amplitude of external oscillation reaches a critical value. And the disturbances created at this critical stage is sub-harmonic in nature, i.e the frequency of surface waves is exactly half of the externally induced oscillation itself. Also, if the instability is characterized separately for each wave-vector $\mathbf{k}$ by it's own critical amplitude a, then the instability boundaries in $a - k$ space resemble the shape of tongues.

2 Fluid Dynamic analysis

The following derivation closely resembles [Kum96], except a few diversions. Consider an incompressible homogeneous liquid of density ρ dynamic viscosity η and surface tension σ subject to a constant gravitational field g and induced oscillatory field $acos(\omega t)$

$$G(t) = g - acos(\omega t)$$

If p and u are the pressure and velocity perturbation fields then the Navier-Stokes and continuity equation for the incompressible case is:

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}) = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{u}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{1}$$

Experimentally at high enough frequency and moderate amplitudes at the surface

$$\frac{\partial \mathbf{u}}{\partial t} >> (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \tag{2}$$

Thus we get the linearized NS equation for the surface:

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{u} \tag{3}$$

Taking curl on both sides of (2) twice.

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} \tag{4}$$

since curl and $\frac{\partial}{\partial t}$ are commutative Using the continuity condition (1)

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{u} = -\nabla^2 \mathbf{u} \tag{5}$$

Then using the commutativity of $curl$ and ∇^2 operators and the same relation as above.

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \nabla^2 \mathbf{u} = -\nabla^2 \nabla^2 \mathbf{u} \tag{6}$$

Considering $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{x}} + w\hat{\mathbf{z}}$ and kinematic viscosity as μ

$$(\frac{\partial}{\partial t} - \mu \nabla^2) \nabla^2 w = 0 \tag{7}$$

If $z = 0$ is the free surface with vertical perturbation field as $\zeta(\mathbf{x}, t)$ where $\mathbf{x} = (x, y)$ then using first order approximation of kinematic boundary condition of the surface[LL87]:

$$\frac{\partial \zeta}{\partial t} = w\Big|_{z=0} \tag{8}$$

[KCD12]The stress tensor for the incompressible fluid is given by:

$$\pi_{ij} = -p\delta_{ij} + \eta(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) \tag{9}$$

Since at the free surface the shear stresses must be 0

$$\pi_{xz} = \pi_{yz} = 0 \quad hence \quad \frac{\partial \epsilon_{xz}}{\partial x} = \frac{\partial \epsilon_{yz}}{\partial y} = 0 \tag{10}$$

Where ϵ_{ij} is strain rate tensor. Then using continuity equation (1) we get

$$(\nabla_H^2 - \frac{\partial^2}{\partial z^2})w = 0 \tag{11}$$

Following the same small order displacement field of free surface, $\mathcal{O}(slope) << \mathcal{O}(curvature)$. Then the normal stress built in $+\hat{\mathbf{z}}$ element at $z = 0$ by surface tension due to curvature and gauge pressure of ζ height of liquid above $z = 0$ is:

$$\pi_{zz}\Big|_{z=0} = -\rho G(t)\zeta + \sigma\nabla_H^2\zeta \tag{12}$$

Where $\nabla_H^2 \quad is \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then the expression for pressure at $z = 0$ using (4) is:

$$p\Big|_{z=0} = 2\eta\frac{\partial w}{\partial z} + \rho G(t)\zeta - \sigma\nabla_H^2\zeta \tag{13}$$

Another expression for the pressure can be derived by taking the horizontal derivative $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ of (3) and using (1)

$$\nabla_H^2 p = (\rho\frac{\partial}{\partial t} - \eta\nabla^2)\frac{\partial w}{\partial z} \tag{14}$$

Using (12) and (13), we obtain the following expression for $z = 0$ surface

$$(\rho\frac{\partial}{\partial t} - \eta\nabla^2)\frac{\partial w}{\partial z}\Big|_{z=0} = 2\eta\nabla_H^2\frac{\partial w}{\partial z}\Big|_{z=0} + \rho G(t)\nabla_H^2\zeta - \sigma\boldsymbol{\nabla}_H^4\zeta \tag{15}$$

Now if we consider that the surface is infinitely large compared to the local variations in the field ζ then we can take it's Fourier transform in space domain such that

$$\zeta(\mathbf{x}, t) = \frac{1}{(2\pi)^2}\int\limits_{\mathbf{k}} \zeta_k(t)e^{\mathbf{k}\cdot\mathbf{x}}dk_x dk_y \tag{16}$$

Similarly we can decompose $w(\mathbf{x}, z, t)$ as follows

$$w(\mathbf{x}, z, t) = \frac{1}{(2\pi)^2} \int_{\mathbf{k}} w_k(z, t) e^{\mathbf{k}\cdot\mathbf{x}} dk_x dk_y \tag{17}$$

The Fourier components $w_k(z, t)$ and $\zeta_k(t)$ are related at $t = 0$ by the kinematic boundary condition (8) Since the operators acting on w in the equation (7), (11) and (15) are linear are are valid for infinite values for $\mathbf{x}$, so the same equations must be satisfied by $w_k(t)e^{\mathbf{k}\cdot\mathbf{x}}$. The laplacian operator transforms as:

$$\nabla^2 w_k(t) exp(\mathbf{k} \cdot \mathbf{x}) = (\frac{\partial^2}{\partial z^2} - k^2) w_k(t) exp(\mathbf{k} \cdot \mathbf{x}) \tag{18}$$

Thus transforming the equations into $(\mathbf{k}, z, t)$ domain we get:

$$[\frac{\partial}{\partial t} - \mu(\frac{\partial^2}{\partial z^2} - k^2)](\frac{\partial^2}{\partial z^2} - k^2) w_k = 0 \tag{19}$$

$$(\frac{\partial}{\partial z^2} + k^2) w_k \bigg|_{z=0} = 0 \tag{20}$$

$$[(\rho\frac{\partial}{\partial t} - \eta\frac{\partial}{\partial z^2} + 3\eta k^2)\frac{\partial w_k}{\partial z}]\bigg|_{z=0} = -[\rho G(t) + \sigma k^2] k^2 \zeta_k \tag{21}$$

All the Fourier components should satisfy the boundary conditions as $(\mathbf{x}, z, t)$ does:

$$w_k \bigg|_{z=-h} = 0 \tag{22}$$

$$\frac{\partial w_k}{\partial z}\bigg|_{z=-h} = 0 \tag{23}$$

And finally the relation between the two spectra using the kinematic boundary condition (8)

$$\frac{\partial \zeta_k}{\partial t} = w_k \bigg|_{z=0} \tag{24}$$

3 Linear Stability Analysis

Since equations (19)-(23) are linear differential equations and the highest order partial derivative w.r.t time is $\frac{\partial}{\partial t}$, thus an equation of the following form:

$$\frac{\partial \zeta_k(t)}{\partial t} = \mathbf{A}(t)\zeta_k(t) \tag{25}$$

where $\mathbf{A}$(t) is some **some linear operator with period** $\frac{2\pi}{\omega}$ since the only time varying term is $G(t)$ and has the same period. Thus according to Floquet theory, the solutions to $\zeta_k(t)$ could be written in the form[KT94]:

$$\zeta_k(t) = e^{\mu t} Z(t) \tag{26}$$

where $Z(t)$ is some periodic function with period $\frac{2\pi}{\omega}$. Thus if we do the Fourier series expansion of $Z(t)$ then $\zeta_k(t)$ could be written as:

$$\zeta_k(t) = e^{\mu t} \sum_{n=-\infty}^{\infty} \zeta_n e^{\iota n \omega t} \tag{27}$$

Similarly, applying Floquet theory and Fourier analysis to $w_k(z,t)$ we get

$$w_k(z,t) = e^{\mu t} \sum_{n=-\infty}^{\infty} w_n(z) e^{\iota n \omega t} \tag{28}$$

Now the Floquet exponent μ a complex number can be expressed as $\mu = s + \iota \alpha \omega$. For every real value of α, the series ζ_n must make $\zeta(t)$ real for all t. For a special case $\alpha = 0$

$$(e^{\mu t} \zeta_n e^{\iota n \omega t})^* = e^{st} \zeta_n e^{-\iota n \omega t}$$

Thus for reality of $\zeta_k(t)$ we have $\zeta_n^* = \zeta_{-n}$. And for another special case $\alpha = \frac{1}{2}$ we have

$$(e^{\mu t} \zeta_n e^{\iota n \omega t})^* = e^{st} \zeta_n e^{-\iota (n+\frac{1}{2}) \omega t}$$

which clearly implies $\zeta_{n-1}^* = \zeta_{-n}$. This 2 special values of α correspond to harmonic and subharmonic solutions. Now using equation (24),(27),(28)

$$w_n(0) = (\mu + \iota n \omega) \zeta_n \tag{29}$$

Now equation (19)-(24) written in $(\mathbf{k}, z, t)$ domain, if transformed into $(\mathbf{k}, z, n)$ then the operator $\frac{\partial}{\partial t}$ transforms according to

$$\frac{\partial e^{\mu t} w_n(z) e^{\iota n \omega t}}{\partial t} = (\mu + \iota n \omega) w_n(z) e^{\mu + \iota n \omega t}$$

Hence the equation (19) transforms to

$$(\frac{\partial}{\partial z^2} - k^2)(\frac{\partial}{\partial z^2} - q_n^2) w_n(z) = 0 \tag{30}$$

Since solutions to equations $(\frac{\partial}{\partial z^2} - c^2)f(z)$ for different complex numbers c are orthogonal, thus the nullspaces of the operators $(\frac{\partial}{\partial z^2} - k^2)$ and $(\frac{\partial}{\partial z^2} - q_n^2)$ are orthogonal. Hence the general solution to equation (30) lies in the function space formed by the direct sum of the nullspaces [HK71]. In other words

$$w_n(z) = P_n cosh(kz) + Q_n sinh(kz) + R_n cosh(q_n z) + S_n sinh(q_n z) \quad (31)$$

Inserting (31) in (20)

$$2k^2 P_n + (q_n^2 + k^2)R_n = 0 \quad (32)$$

And using (29) we get

$$P_n + R_n = \mu(q_n^2 - k^2)\zeta_n \quad (33)$$

Solving for P_n and Q_n using (31) and (32)

$$R_n = -2\mu k^2 \zeta_n \quad (34)$$

$$P_n = \mu(q_n^2 + k^2)\zeta_n \quad (35)$$

Inserting (34) and (35) into boundary conditions (22) and (23) we get

$$S_n = -\frac{[kP_n + R_n(kcosh(q_n h)cosh(kh) - q_n sinh(q_n h)sinh(kh)]}{[q_n cosh(q_n h)sinh(kh) - ksinh(q_n h)cosh(kh)]} \quad (36)$$

$$Q_n = R_n[q_n sinh(q_n h)cosh(kh) - kcosh(q_n h)cosh(kh)] - \quad (37)$$
$$S_n[q_n cosh(q_n h)cosh(kh) - ksinh(q_n h)sinh(kh)]$$

Inserting (31) in (21) and using (34)-(37) we get the following recursive relation

$$A_n \zeta_n = a(\zeta_{n-1} + \zeta_{n+1}) \quad (38)$$

Where the factor $A_n$is givem by

$$A_n = \frac{2}{k}[gk + \frac{\sigma}{\rho}k^3 - \mu^2(\frac{4q_n k^2(q_n^2 + k^2) - C_c osh(q_n h)cosh(kh) + D_n sinh(q_n h)sinh(kh)}{q_n cosh(q_n h)sinh(kh) - ksinh(q_n h)cosh(kh)})] \quad (39)$$

with

$$C_n = q_n(q_n^4 + 2q_n^2 k^2 + 5k^4) \quad (40)$$

$$D_n = k(q_n^4 + 6q_n^2 k^2 + 5k^4) \quad (41)$$

Hence we obtain a recursive relation for the Fourier components for the function $Z(t)$ for particular amplitude of external oscillation a and the wave-number k of the surface wave. This recursive relation when solved **exactly** would give the solution.

4 Matrix formulation of recursive relation

The recursion relation obtained in the previous section can be represented into matrix form in the Fourier components ζ_n is represented as a vector ζ. The suitable matrix equation would then be

$$\begin{bmatrix} \ddots & & & & & & \\ & A_{-2} & 0 & 0 & 0 & 0 & \\ & 0 & A_{-1} & 0 & 0 & 0 & \\ & 0 & 0 & A_0 & 0 & 0 & \\ & 0 & 0 & 0 & A_1 & 0 & \\ & 0 & 0 & 0 & 0 & A_2 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \zeta_{-2} \\ \zeta_{-1} \\ \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{bmatrix} = a \begin{bmatrix} \ddots & & & & & & \\ & 0 & 1 & 0 & 0 & 0 & \\ & 1 & 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & 1 & 0 & \\ & 0 & 0 & 1 & 0 & 1 & \\ & 0 & 0 & 0 & 1 & 0 & \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \zeta_{-2} \\ \zeta_{-1} \\ \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{bmatrix} \tag{42}$$

Or equivalently

$$A\zeta = aB\zeta$$

Since A is a diagonal, it's inverse is readily know hence equation (42) can be written as an eigenvalue problem as

$$A^{-1}B\zeta = \frac{1}{a}\zeta \tag{43}$$

Now the elements A_n depend on the s, k, ω. Now consider the evolution of $\zeta_k(t)$ with time which has been decomposed into spectrum of functions of the form

$$e^{st+\iota((n+\alpha)\omega t}$$

where non-oscillatory part is e^{st} with s being real. If $s > 0$ then the expression blows up to infinity with time. On the contrary is $s < 0$ then expression approaches 0 or stays 0 if no initial disturbance is caused. Thus the stability boundaries are characterized as $s = 0$. So now for computing the elements of matrix A if we fix $s = 0$ and k to any value then we get infinite number of eigenvalues $\frac{1}{a}$ which lead to the specific series of amplitude for which the waves with wave-number k are stable and the largest eigenvalue corresponds to the minimum required amplitude for that particular k to emerge into instability zone. Thus if we plot such amplitudes w.r.t k we may get a minima of those amplitudes which represent the the critical amplitude below which no standing wave is excited. To numerically compute the eigenvalues for the subharmonic excitation, we truncate the matrix in such a way that we get the

Fourier components of $Z(t)$ till n=4 and in order to obtain real-components of computed eigenvectors using the condition $\zeta^*_{n-1} = \zeta_{-n}$ we would need the values ζ_n with $n \in [-5, 4]$, hence we need to truncate the following 10×10 matrix from A^{-1}. The matrix is computed as follows:

for i in [1,10]:
 if i=1:
  $[A^{-1}B]_{i+1,i}$
 else if i=10:
  $[A^{-1}B]_{i-1,i}$
 else if i<6:
  $[A^{-1}B]_{i+1,i} = [A^{-1}B]_{i-1,i} = A^*_{5-i}$
 else:
  $[A^{-1}B]_{i+1,i} = [A^{-1}B]_{i-1,i} = A_{i-6}$

Analytically, the infinite dimensional matrix $[A^{-1}B]$ would give real eigenvalues, but proper 10×10 gives nearly real eigenvalues for small values of k, i.e the $\left|real(\frac{1}{a})\right|$ exceeds $\left|imaginary(\frac{1}{a})\right|$ at least by a factor 100 for small values of k. But for relatively higher values of k, some phase of some complex approach $\pm 45\,\mathrm{deg}$, which are not physical since a needs to attain real values. Thus ignoring such eigenvalues we sort the rest in order of absolute values of real part and plot the inverses of first 4 largest ones. For the following parameters used:

Density: 1.2 $gm \quad cm^{-3}$
Surface Tension: 0.0676 N/m
Dynamic Viscosity Coefficient: $1.02 \times 10^{-4} Pa - sec$
Frequency: 60 Hz
g: 9.81 ms^{-1}

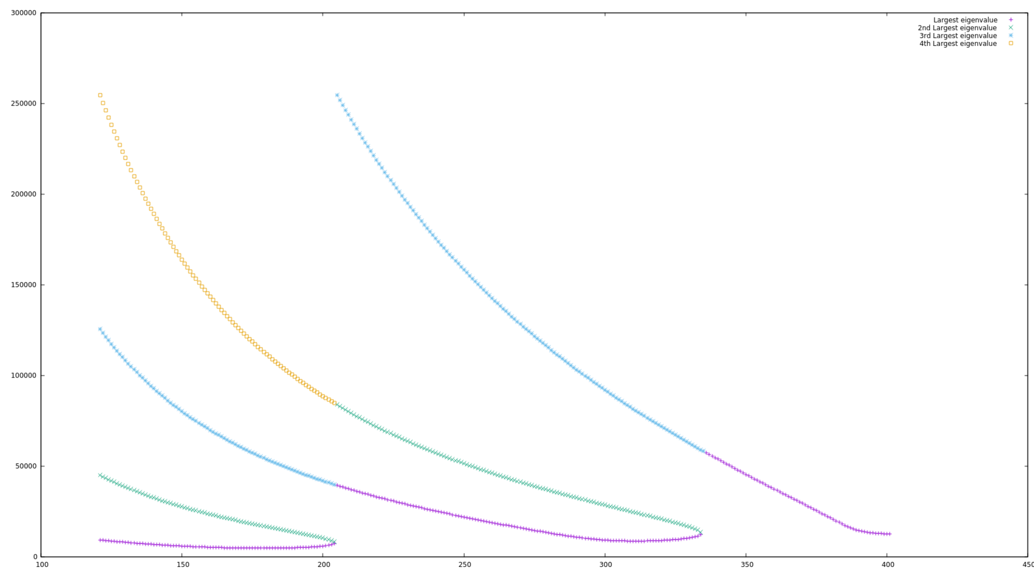

Figure 1: y-axis: $a(m)$, x-axis: $k(m)$

These are the stability zones that we obtain. Corresponding to the curve for inverse of the largest eigenvalue, we see that there exists a local minima. That corresponds to the critical value of amplitude of external oscillation below which no standing wave is excited. The critical amplitudes for each value of applied frequency is computed using gradient descent over the inverse of largest eigenvalues

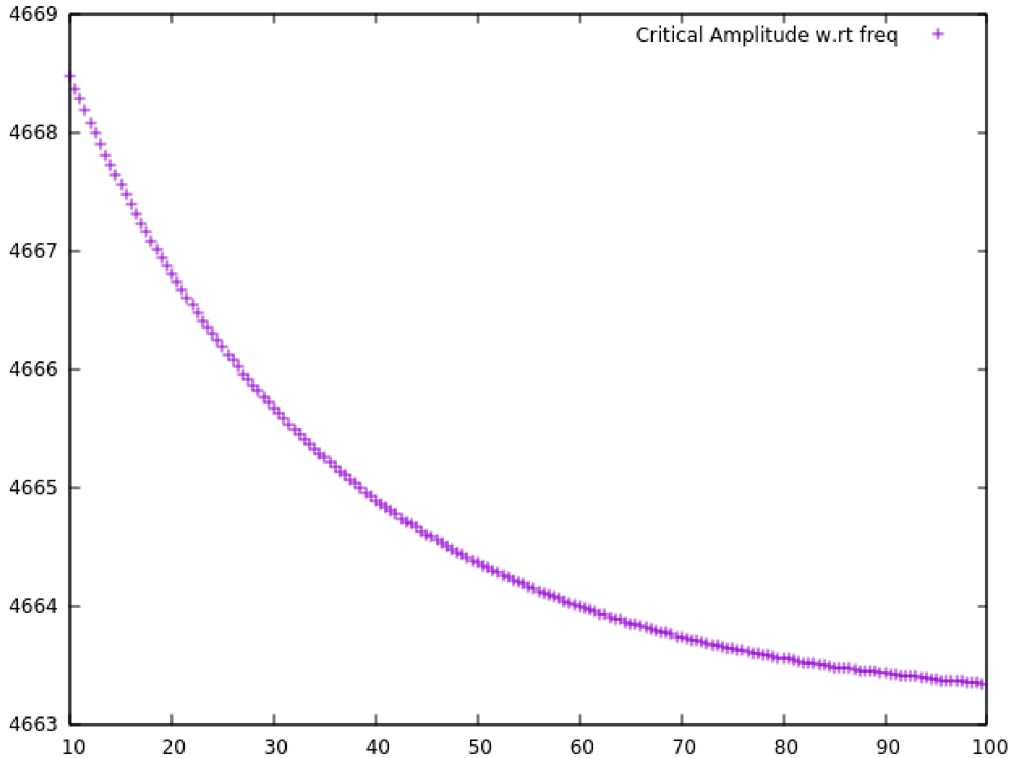

Figure 2: y-axis: a, x-axis: Frequency

Then obtain curve in Figure 2 which shows an approximately hyperbolic decrease in the critical amplitude with applied frequency.

In order to mathematically model the decrease, we can approximate it with a function of form $y = \frac{C}{x^{\alpha+1}}$ using the following property

$$\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} = -\frac{x}{\alpha+1} = kx \tag{44}$$

For a labelled 1D data $x[i], y[i]$, we can re-sample the data using (44) and discretizing the derivatives:

$$Y[i] = \frac{y[i+1] - y[i-1]}{y[i+1] + y[i-1] - 2y[i]} = kx[i] \tag{45}$$

It thus reduces to convex optimization of squared error loss function and α could be computed. Further computing C is straightforward.

5 Conclusion

Understanding it physically, when a column of viscous fluid is oscillated vertically, the surface remains stable and unperturbed, until some additional slight perturbation is made in some part of the surface. This disturbance can be decomposed into sinusoidal waves and hence it collectively represents a a bunch of wave-numbers. If the amplitude of oscillation is bellow the critical amplitude, all the sinusoidal waves(characterized by an amplitude and wave-number) will **damp out exponentially** and the surface tends to stability again, since below critical frequency, any sinusoidal wave(with any wave number) is stable as we see in Figure 1.
But if the amplitude of oscillation is even slightly greater than critical frequency, then if we create a small(and spatially asymmetric) perturbation in any part of the surface, since the Fourier transform suggests it is composed of all infinite band of sinusoidal waves(all possible values of k), there exists a range of such waves lying in the first instability zone according to Figure 1. **These particular waves will blow up exponentially** until the linearization made in Equation (2) is invalid. Also the critical amplitude decreases with the applied frequency.

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