General Theory of Relativity
Term Project

SHADOWS OF BLACK HOLES AND RENDERING AN IMAGE OF ACCRETION DISK UNDER GRAVITATIONAL LENSING

Shobhit Saheb Dey
Guided By - Prof. Sayan Kar
Department of Physics
Indian Institute of Technology Kharagpur

Introduction

When black holes are observed using any of imaging techniques, there is always a dark region often *incorrectly* labeled as the event horizon. This is related to the fact that light emitted at certain distances away from the event horizon, at a certain ranges of angle do fall back into the horizon. And this leads to the dark region in the images actually projecting a larger circle(for a non-rotating black hole) than the event horizon and the region is labeled the shadow. In this project, some properties of null geodesics are studied based on conserved quantities following [Syn66] and then a different set of equations for geodesics is derived, convenient to numerical simulations and ray-tracing for generating a virtual image. Ray tracing technique to render an image of black hole with an accretion disk is also used in the last section.

Null Geodesics in Schwarzschild metric

Since we are going to study all the effects on null geodesics from distant imaging point of view, it is feasible to use Schwarzschild coordinates (t, r, θ, ϕ) where all of them have usual physical interpretation except r is not the actually the *radial distance* but defined as $r = \sqrt{A/4\pi}$ where A is the total spherical area at r. The line element in the Schwarzschild metric goes as:

$$ds^2 = -(1 - R/r)dt^2 + (1 - R/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

From the isometries of this metric, 2 of the Killing Vectors are $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ constraining the motion in $\theta = \pi/2$ plane. Since for any Killing vector K, $K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ is a conserved quantity along the geodesic $x^\mu(\lambda)$. Hence 2 of the conserved quantities in Schwarzschild metric are:

$$(1 - R/r)\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E \tag{2a}$$

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = L \tag{2b}$$

If $\beta = E/L$, also a conserved quantity, then from these 2 equations we get:

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \frac{\beta r^2}{1 - R/r} \tag{3}$$

For null geodesics $ds^2 = 0$, so using (1) and (3) we get:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = r^4\left(\beta^2 - \frac{r-R}{r^3}\right) = r^4 F(r) \tag{4}$$

Although we have a differential equation for the null geodesic, this is not suitable for numerical simulation because $F(r)$ can turn negative, not necessarily due to non-physical geodesic coordinate, but due to discretization of numerical scheme. The angle at which the beam is released, ψ can be computed from an infinitesimal triangle of vertices $(r, \pi/2, \phi)$, $(r + dr, \pi/2, \phi)$, $(r, \pi/2, \phi + d\phi)$ using the line element from Eq(1):

$$\cot^2\psi = \frac{1}{r^2(1-R/r)}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 \tag{5}$$

And then using Eq(4) we get:

$$\cot^2\psi = \frac{r^3\beta^2}{r-R} - 1 \tag{6}$$

Now since β is a constant along the geodesic, after some rearrangement we get a constant as a function of r and ϕ which can be determined by the initial variables r_0 and ϕ_0:

$$\beta^2 = \frac{r-R}{r^3\sin^2\psi} = \frac{r_0 - R}{r_0^3\sin^2\psi_0} \tag{7}$$

Condition for the escape of photons

Consider a beam that was emitted in some angle ψ at some radial coordinate r. Now if the beam is propagating outwards such that $\frac{\mathrm{d}r}{\mathrm{d}\phi} > 0$, then for some point along the geodesic $F(r)$ can become 0 i.e $\frac{\mathrm{d}r}{\mathrm{d}\phi} = 0$. Before this point one of the 3 situations can occur. 1. The beam can go forward and r increases further, 2. The beam can revolve indefinitely at the same r or 3. it can spiral inwards. If the first situation were to occur, $F(r)$ becomes negative and this isn't physically allowed by Eq(4). The second situation would imply a non-differentiability of 3rd order. Thus the only possibility is that it spirals inwards. Thus for any outward propagating beam, if along the geodesic a critical point is obtained, it is going to spiral down.

In Figure 1, consider the 3 beams initiated outside a black hole with $R = 1$ at (r, β^2) coordinates(completely specifying (r, ψ)), all initiated in outwards direction ($\frac{\mathrm{d}r}{\mathrm{d}\lambda} > 0$). Since β^2 is constant, they move horizontally in this coordinate space. So beam A hits the boundary curve where $\frac{\mathrm{d}r}{\mathrm{d}\phi} = 0$ and then starts falling inwards as explained earlier. While beam B being in the same radial coordinate but initiated at an angle of higher β^2 i.e lower ψ never hits the

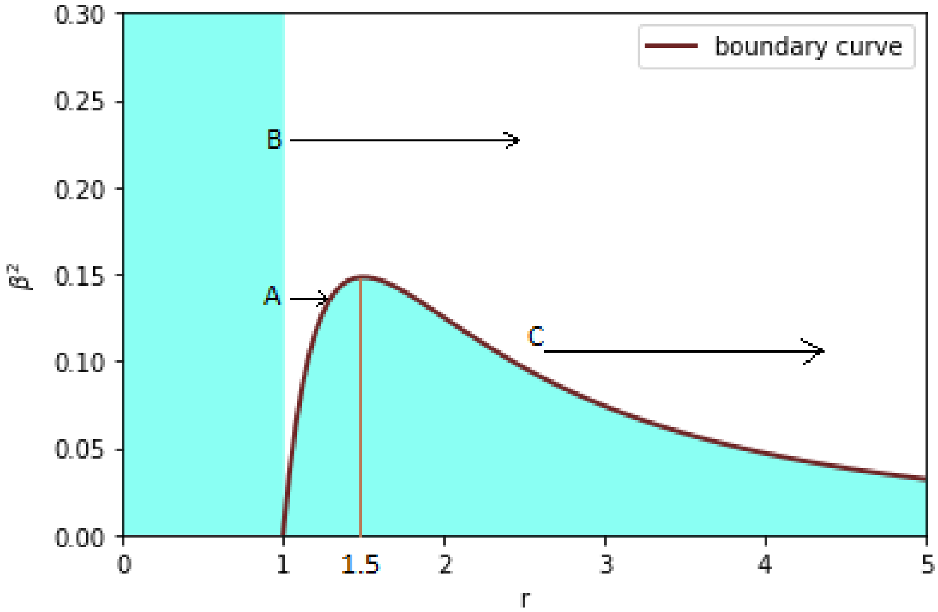


Figure 1: Boundary curve for apsis of light and the shaded regions show inaccessible regions

boundary curve and escapes to infinity. Beam C does the same with lower β^2 hence larger ψ but at a larger radial coordinate. The shaded region in the left is inside event horizon while below boundary curve region is inaccessible because $\sin^2\psi \in [0,1]$. This analysis implies that for $r > 3R/2$, the light beam escapes to infinity to any value of β^2 corresponding to $\psi \in (0, \pi/2)$ but for $r < 3R/2$ the beam escapes to infinity only if the value of β^2 is above the maxima of the boundary curve that is $4/27R^3$. Thus the beams only inside a cone escape to infinity and the cone opens up fully in the region $r > 3R/2$.

Simulating geodesics for ray-tracing

As mentioned earlier, Eq (4) can't be used to simulate the null geodesics because $F(r)$ can take negative values along the geodesic and if not, discretization may lead to the same. We make a change of variable $u = 1/r$ and then Eq (4) becomes:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2(1 - Ru) = \beta^2 \tag{8}$$

The form of this equation is similar to that of energy conservation equation in Newtonian dynamics: $T + V = E$ where $T = \frac{m}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2$ with $m = 2$ having no physical significance, $V = u^2(1 - Ru)$ and $E = \beta^2$ also a constant of motion here. Thus, similar to Newtonian mechanics, we use $m\frac{d^2u}{d\phi^2} = -\frac{dV}{d\phi}$ to get:

$$\frac{d^2u}{d\phi^2} = \frac{3Ru^2}{2} - u \tag{9}$$

Through simulation it shall be demonstrated that a ray with $\beta^2 < 4/27R^3$ escape to infinity no matter at what radial coordinate was it initiated or passes through. Meanwhile rays with $\beta^2 > 4/27R^3$ enter the photon sphere and spirals into the event horizon. Reason being that if β^2 is larger than the maxima of $u^2(1-Ru)$, then $\left(\frac{du}{d\phi}\right)^2$ is never negative and hence u can attain all the values from $(0,\infty)$ allowing the beam to spiral in. But if β^2 is less than the maxima, the values of u are constrained. The beams will be first assumed to be initiated at infinity towards the black hole, all parallel to each other, mimicking to ray-tracing technique used when the camera is at a large distance($r >> R$) from the black hole. But to simulate Eq (10), 2 initial conditions are needed. One is obviously $u = 0$ since $r -> \infty$ and the other we get from Eq (9), plugging $u = 0$ and condition that $\left(\frac{du}{d\phi}\right)_{\lambda=0} > 0$ since the ray is approaching the black hole, we get $v_0 = \left(\frac{du}{d\phi}\right)_{\lambda=0} = \beta$. So we get a set of coupled linear ODEs:

$$\frac{du}{d\phi} = v \tag{10a}$$

$$\frac{dv}{d\phi} = \frac{3Ru^2}{2} - u \tag{10b}$$

With the initial conditions $u = 0$, $v = \beta$ at $\phi = 0$. To interpret the visuals of the simulation, the physical significance of β should be found. By definition it is E/L. And in $r -> \infty$, the spacetime is asymptotically Minkowskian. By convention the four-momentum along a null geodesic in Minkowskian space is defined as $\frac{dx^\mu}{d\lambda}$ i.e choosing the affine parameter such that $\frac{dx^0}{d\lambda} = E$. Then correspondingly momentum and angular momentum can be quantified. Making the identification $L = |\mathbf{r} \times \mathbf{p}|$, for a null geodesic we get:

$$\beta^2 = \left(\frac{E}{L}\right)^2 = \frac{1}{b^2} \tag{11}$$

Where b is the impact parameter of the null geodesic. **Thus the impact parameter indirectly characterizes the escape or spiralling-in of the beam.** Now with all the initial conditions and the parameters, the null geodesic is simulated by using 4^{th}-order Runge-Kutta scheme on Eq(11a) and Eq(11b). The result for different values of $v_0 = \beta$ are shown in Figure 2. The chosen value of Schwarzschild radius(R) is 1. It can be seen that for beams with β greater than the critical value spiral into the horizon. The beam with β_c should have infinitely revolved

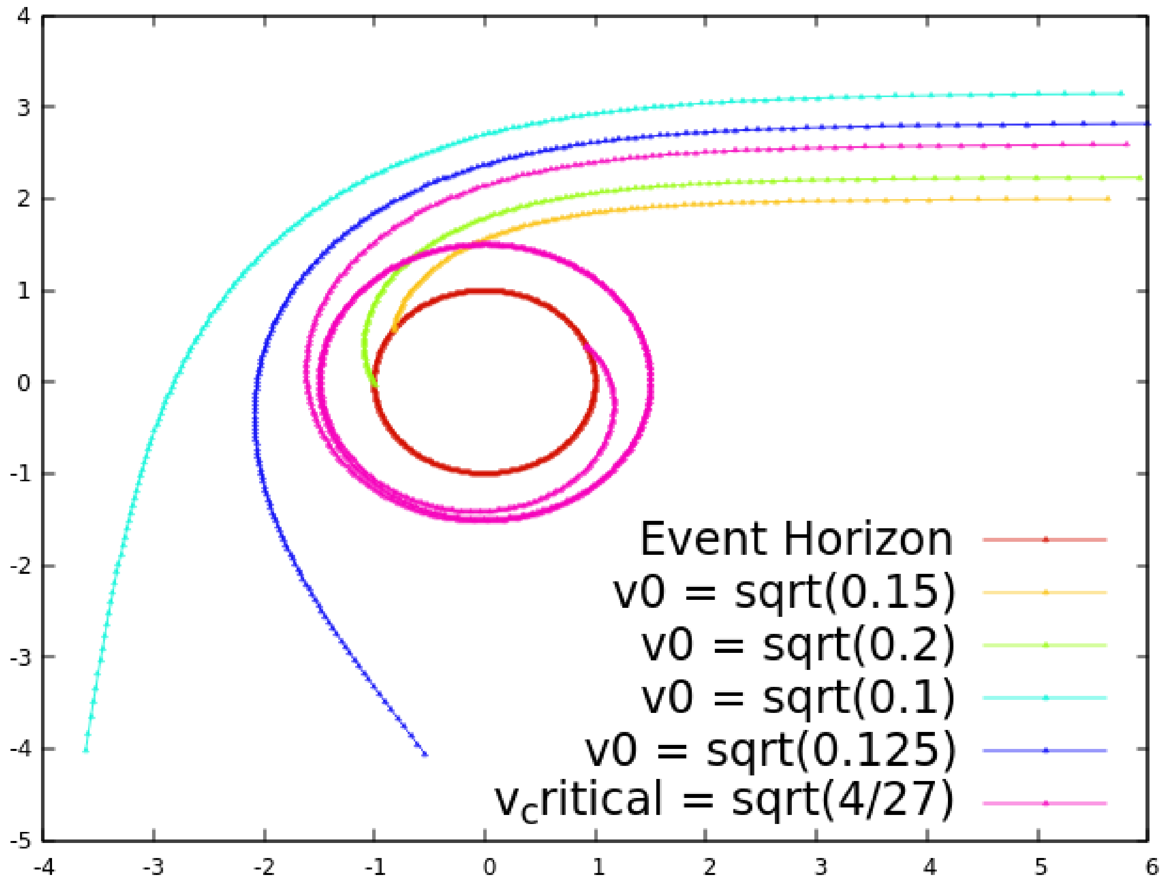


Figure 2: Null geodesics with different impact parameters

in the photon sphere but it eventually spirals in after 1 revolution due to numerical error of discretization and machine precision. The beams with β greater than the critical value escape to infinity. A similar characteristic is observed if the beams were initiated at some finite distance. The result for this simulation is shown in Figure 2. Here again $R = 1$ and the radial coordinate where beams are initiated is $r_0 = 4$. For any chosen value of β, the initial conditions are $u_0 = 1/r$ and using Eq (9), $v_0 = \sqrt{\beta^2 - u^2(1-u)}$. Using these, beams with different β are simulated and it is again seen that those with higher than the critical value of β spiral into the horizon. Now turning towards the relation of these geodesics to image generation. Since the form of the geodesic equations using the Christoffel connection in Schwarzschild metric is invariant under $\lambda - > -\lambda$[Car19]. This means that the rays will be retraced to the emission point if some other emitter is situated along the same geodesic. Since all the monocular imaging techniques generate only the projection of rays onto the sensor plan, only the angular position of emitters will be known. Now from Figure 3 it can be seen that if a camera at $(r = 4, \phi = 0)$ points to center, then only the rays with $\beta^2 < 4/27$ reach

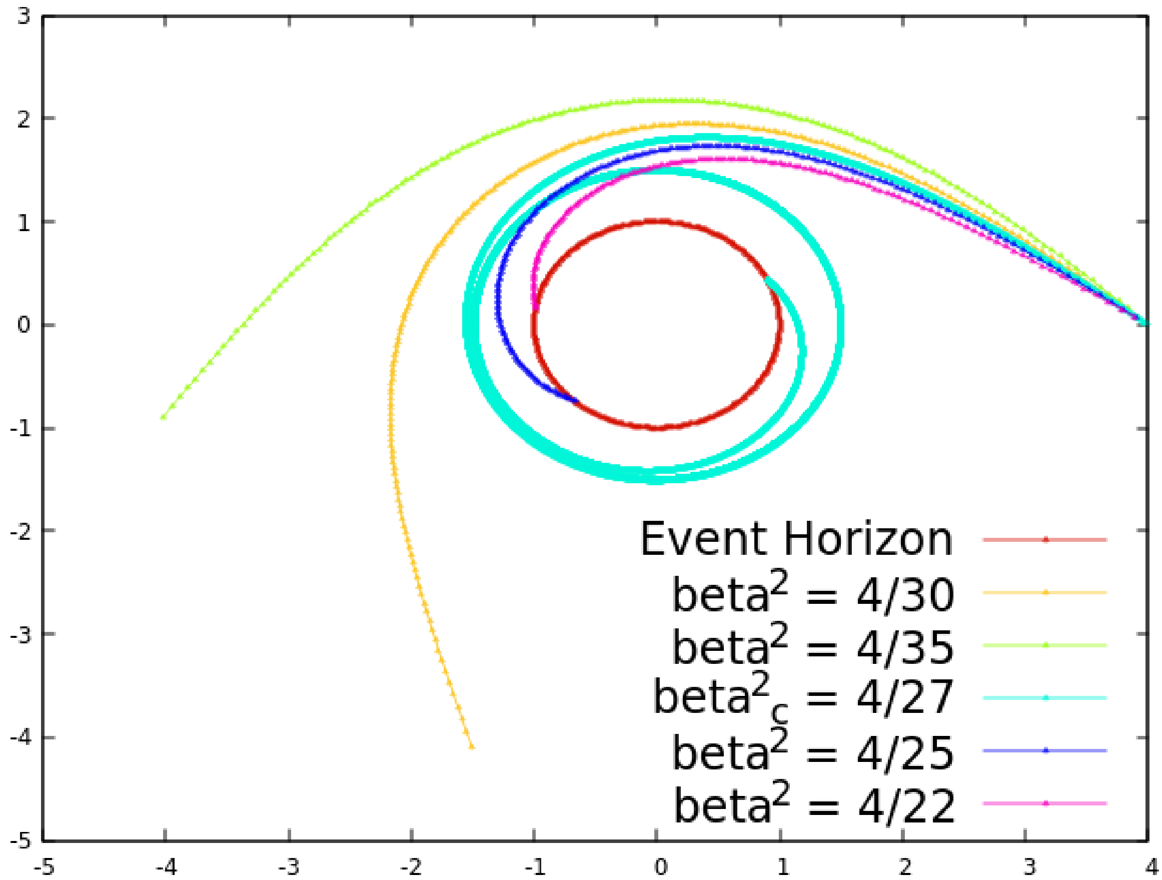


Figure 3: Null geodesics with different β from a finite distance

the sensor plane from emitters at infinity. In astrophysical situations these correspond to all the light sources/reflectors(stars, planets etc.) in the background. But all these beams when reach the camera, they are incident on an angle(using Eq (7)):

$$\psi = \arcsin \frac{1}{\beta} \sqrt{\frac{r-R}{r^3}} \tag{12}$$

And all the rays that initiated far away from the photon sphere have β less than the critical value. So the largest value of ψ is computed from $\beta_{critical} = \sqrt{4/27R^3}$ is:

$$\psi_{min} = \arcsin \sqrt{\frac{R^3(r-R)}{r^3}} \tag{13}$$

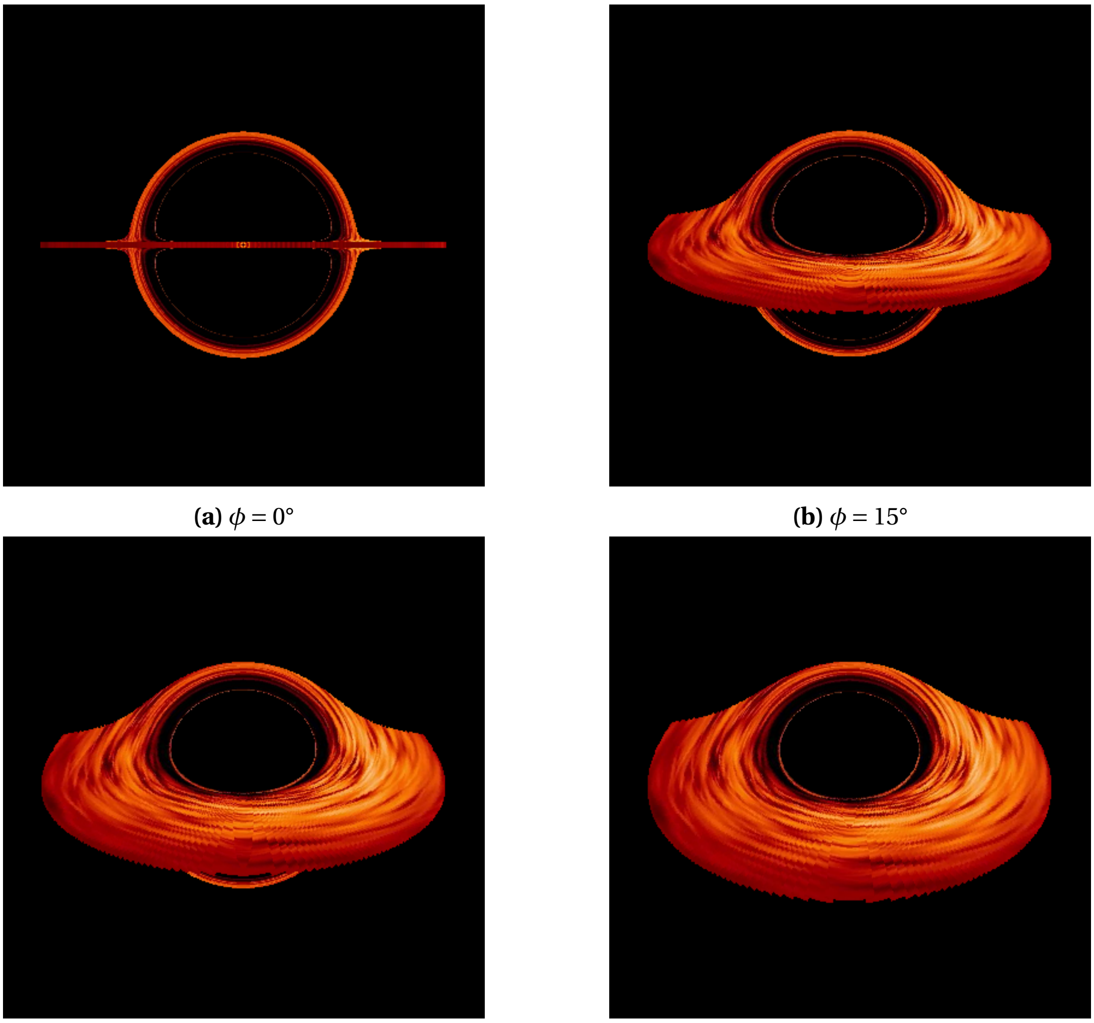
All the rays within the $2\psi_{min}$ cone are traced back to emitters either nearer than the black hole or those within the photon sphere. Assuming there aren't any emitters there(the case for accretion disk will be taken up in the next section), the observer is not going to sense any light in this angular region. And this region is called the 'shadow of the black hole'. This region in the image is usually a black circular region in the image and hence the name.

RENDERING AN IMAGE OF A BLACK HOLE WITH ACCRETION DISK

The image rendering technique to be used here has a simple procedure though computationally very intensive. Every pixel in the camera corresponds to an angle at which a ray reaches the pinhole/aperture. We first need initialize an angular width of view(ψ_{max}) and a pixel resolution(n_p). Using these we compute a focal length as $f = \arctan\left(n_p/\psi_{max}\right)$. Now these quantities aren't actually practical for the available telescopes but aren't wrong for numerical rendering. We assume that the camera's plane is normal to the radial direction that projects onto the midpoint of the camera's plane. So for every pixel we initiate a beam with $u_0 = 1/r_{camera}$ and use the pixel's direction from focus, i.e ψ, to compute β using Eq (7) and hence v_0 using Eq (9).
Now we require a 3D map of emitters(position, intensity, color). Ideally, the GMHRD simulations for the accretion disk should be run and a 3D map should be used but those simulations require high compute(parallelization) and storage capacity. Even if a 3D map is available, due to the non-linearity it is needed to be simulated in high resolution. Now if we are generating an image of 400×400 then 160000 null geodesics need to be simulated and 400 is already not a good resolution. So due to compute resource constraints, we consider a simple technique were we take a pre-simulated image of the accretion disk from the top-view and without any lensing and then render the same image with the ray-tracing technique according to different camera coordinates. The image is first mapped to $r = 1.5$ to $r = 4$ in the x-z plane. Now if any of the null geodesic enter this annular region, then the corresponding pixel is colored. A suitable thickness is required because the RK4 scheme might just skip the region due to discretization and that would lead to some false negatives in the image. The parameters set for the camera were:

- Angular resolution = 60°
- Pixel resolution = 400×400
- Radial coordinate = 10
- Azimuthal position = 0°, 15°, 22.5°, 30°

With these technique and initialization, the following images were rendered on C++ code:



(a) $\phi = 0°$ **(b)** $\phi = 15°$

(c) $\phi = 22.5°$ **(d)** $\phi = 30°$

Figure 4: Images rendered using ray-tracing from different azimuths

Conclusion

From the 1st and 2nd section it was clear that photons emitted at certain distance withing the photon sphere escape to infinity only if emitted in a cone of particular angle. Then in the next sections, the conservation equations were modified for convenient numerical simulation of null geodesics showing that rays incident towards the black hole spiral inwards for the values of β higher than the critical. And those light rays coming from infinity reach an observer outside a certain angular region given by Eq(14). Finally a ray-tracing technique was used to image a black hole with an accretion disk in the last section.

Bibliography

[Syn66] JL Synge. "The escape of photons from gravitationally intense stars". In: *Monthly Notices of the Royal Astronomical Society* 131.3 (1966), pp. 463–466.

[Car19] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019.