ELECTRODYNAMICS-2
TERM ESSAY

RELATIVISTIC EVOLUTION OF SYNCHROTRON RADIATION

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INTRODUCTION

As an immediate consequence of Lienard-Wiechert fields, accelerating point charges emit radiation. In a special regime of accelerated motions, we consider a point charge moving in a circular motion in presence of a magnetic field(called the cyclotron or synchrotron, varying in speed). At non-relativistic speeds, the radiation pattern is that of cyclotron regime and with increasing speeds the radiation pattern transits to synchrotron regime. The analysis of synchrotron radiation directly from Lienard-Wiechert fields in lab frame is algebraically tedious and **conceals the more physical interpretation of cyclotron to synchrotron transition** i.e the transition to different attributes of emitted radiation, with increasing speed. To obtain a more physical understanding of the various transition effects we study the radiation in an frame where the particle is only momentarily at rest. We then Lorentz-transform to lab frame which phenomenologically gives rive to **relativistic beaming** explaining by the order of relativistic beaming.

CYCLOTRON IN PARTICLE FRAME

Consider a particle moving a speed **v** in a uniform magnetic field perpendicular to the velocity. Consider an *inertial* reference frame S' where the particle is momentarily at rest and at the origin. Due to Lorentz transformation of electromagnetic fields, magnetic field in lab frame B gives rise to electric field $E' = \gamma_v vB$ that accelerates the charge(viewed as centripetal acceleration in lab frame) and a magnetic field $B' = \gamma_v B$ which provides no force since the particle is at rest in this frame. If the speed of particle in S' is u then the force law gives:

$$\gamma_u m\dot{u} + \gamma_u^3 (\frac{u}{c})^2 \dot{u} = F' = qvB\gamma_v \tag{1}$$

Momentarily rest condition gives $u = 0$ and $\gamma_u = 1$ then :

$$a = \dot{u} = \frac{qvB\gamma_v}{m} \tag{2}$$

This instantaneous acceleration from rest in S' is studied using Lienard-Wiechert fields at zero velocity and finite acceleration[Zan13]:

$$\mathbf{E_a} = \frac{q}{4\pi\epsilon_0}(\frac{\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a})}{Rc^2})_{ret} \tag{3a}$$

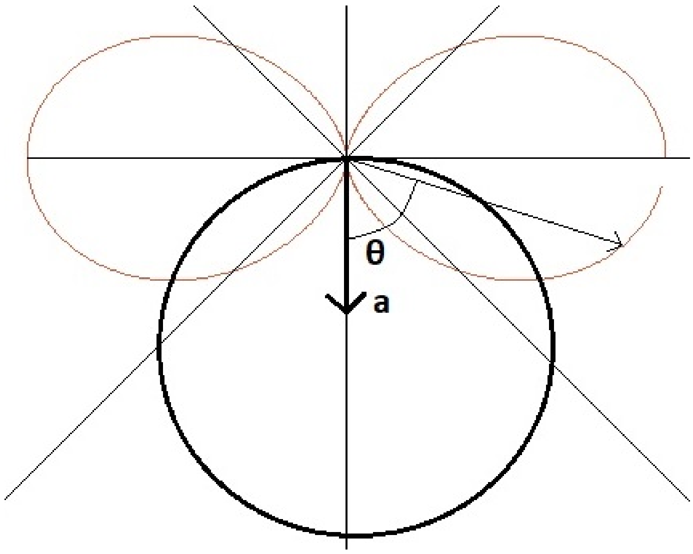

Figure 1: Angular resolution of power

$$\mathbf{B_a} = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{a}\times\hat{\mathbf{R}}}{cR}\right)_{ret} \tag{3b}$$

Since the factor appearing in original equations $k = 1 - \frac{\hat{\mathbf{R}}\cdot\mathbf{u}}{c}$ is suppressed to 1 since $u = 0$, so the detected and emitted power per unit solid angle both are given by:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{R^2}{\mu_0 c}|\mathbf{E_a}|^2 \tag{4}$$

Aligning the z-axis of a right handed coordinate system with **a** and using spherical polar coordinates with Eq. (3a) and Eq(4):

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2 a^2 \sin^2(\theta)}{16\pi^2\epsilon_0 c^3} \tag{5}$$

Angular Resolution of power is shown in Figure (1), where the dumbbell closely resembles the surface of constant power(surfaces were $\frac{1}{R^2}\frac{\mathrm{d}P}{\mathrm{d}\Omega}$ is constant, hence $R = \pm\sin(\theta)$). The 3D shape will be azimuthally symmetric production of the dumbbell. To study the radiation in the frequency domain we make use of the fact that the system retains it's configuration in intervals of $\frac{2\pi}{\omega}$, the angular frequency of the emitted fields must be integral harmonics of ω. In ideal conditions(true homogeneity of magnetic field and no collisions), the radiation spectrum is heavily spiked *only* at ω[Cai12], i.e also the angular frequency of rotation of the particle given by $\frac{qB\gamma_v}{m}$ from Eq. (2). Thus ideally, the cyclotron radiation is monochromatic. When we Lorentz transform to lab frame, this attribute of being monochromatic is broken.

The radiation is going blue and red shifted.

Relativistic beaming of cyclotron radiation

We can decompose the cyclotron radiation into plane waves components and given the dispersion relation $k = \frac{\omega}{c}$ and monochromaticity constraint we can interpret the angular distribution of power $\frac{dP}{d\Omega}$ as a quantity directly proportional to the number of photons of wavenumber $\frac{\omega}{c}$, ω being the angular frequency of particle. To study the observational changes of the photons in lab frame, consider the equation of propagation of fields proportional to $\exp\{i(\mathbf{k}\cdot\mathbf{x}-\omega t\}$ satisfying the source free Maxwell's equation:

$$\delta_\mu \delta^\mu \mathbf{A} = 0 \tag{6a}$$

$$\delta_\mu \delta^\mu \phi = 0 \tag{6b}$$

From Lorentz invariance of the operator $\delta_\mu \delta^\mu$, the quantity $|\mathbf{k}|^2 - (\frac{\omega}{c})^2$ is also Lorentz invariant i.e 0. This suggests that the tuple $K^\mu = (\frac{\omega}{c}, \mathbf{k})$ transforms as a 4-vector and the phase angle $K^\mu X_\mu$ is also invariant.

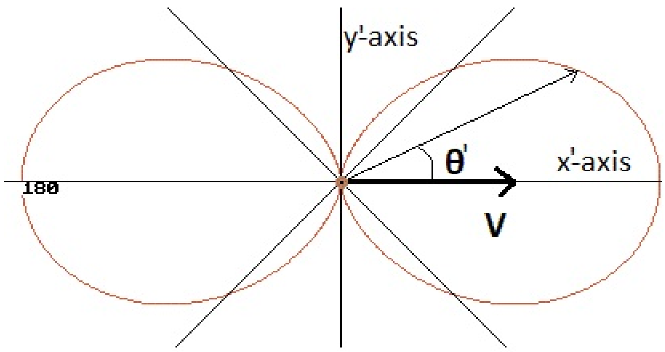

Figure 2: Direction of a photon in S' frame

Consider a photon traveling in a direction θ' from x-axis in the S' frame. Let the wave vector be $k'(\cos\theta', \sin\theta')$ and angular frequency be ω'. The corresponding quantities in lab frame(S) are $k(\cos\theta, \sin\theta)$ and ω. Then using Lorentz transformation generated by velocity of (S' w.r.t S as v in x direction we get:

$$\omega = \gamma_v(\omega' + \beta_v k \cos\theta') \tag{7a}$$

$$k\cos\theta = \gamma_v(k\cos\theta' + \beta_v \frac{\omega'}{c}) \tag{7b}$$

Using the dispersion relation $k = \frac{\omega}{c}$ and $k' = \frac{\omega'}{c}$ ans some manipulation we get:

$$\omega = \gamma_v \omega'(1 + \beta_v \cos\theta') \tag{8a}$$

$$\cos\theta = \frac{\beta_v + \cos\theta'}{1 + \beta_v \cos\theta'} \tag{8b}$$

This pair of equations explains and generates literally all the effects of cyclotron to synchrotron transition. Eq. (8a) shows that the monochromatic radiation in cyclotron radiation is no longer so in the synchrotron regime. A photon travelling in θ' in S' either blue shifted or red shifted by a factor $\gamma_v(1 + \beta_v \cos\theta')$.The photons emitted in the region $\theta' \in (-\frac{\pi}{2}, \frac{\pi}{2})$ are blue shifted while the rest are red shifted. The same photon that was travelling in direction θ' in S' frame is not seen to be travelling in θ in lab frame and the relation between θ' and θ is given by Eq. (8b).

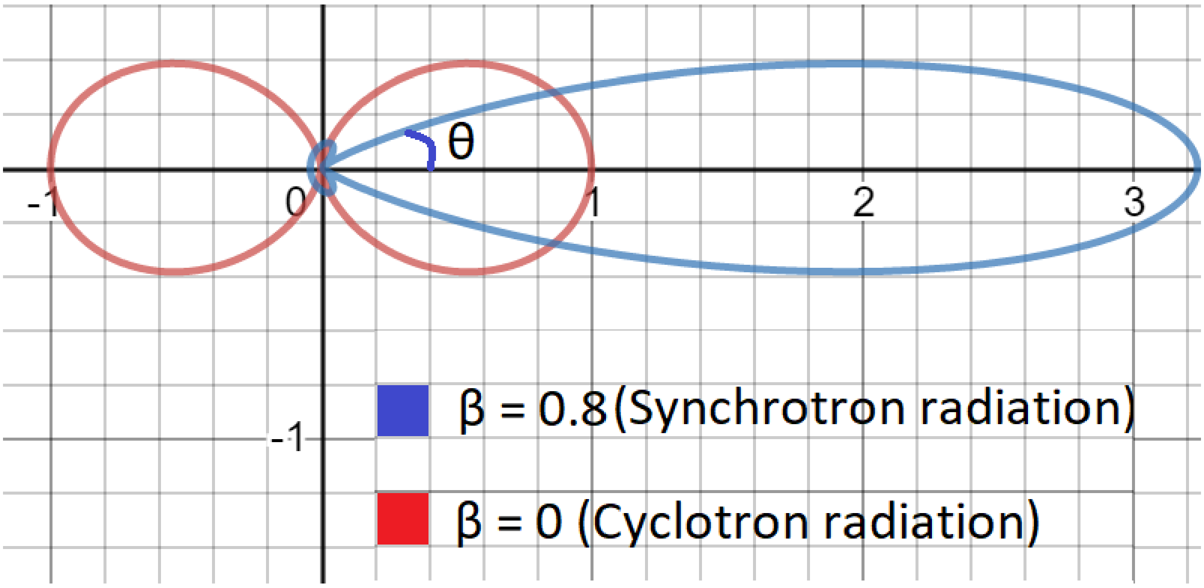

Figure 3: Caption

Now to find the angular power distribution(constraining in xy plane), first note that intensity and $\frac{dP}{d\Omega}$ for any radiation is proportional to $n\omega^2$ where n = no. density of photons. Now in our considered scenario of relativistic beaming, the no. density of photons emitted per unit solid angle in the direction θ' in S' frame is exactly equal to that in direction 4 in S frame, while θ and θ' are related by Eq. (8b). Because all the photons around a θ' are beamed identically. But the frequency of the same neighbourhood of photons is different in S frame, governed by Eq. (8b). So using the correction factor $\frac{\omega}{\omega'}$, expressing θ' in terms of θ and using power law Eq. (5)(note that there θ was the angle from **a** while here it is from x-axis), we get the new power law for the xy plane:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = (\frac{\omega}{\omega'})^2 \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \cos^2\theta' \tag{9a}$$

$$\cos\theta' = \frac{\cos\theta - \beta_v}{1 - \beta_v \cos\theta} \tag{9b}$$

using Eq. (7a) we get:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \gamma_v^2 (1 + \beta_v \cos\theta')^2 \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \cos^2\theta' \tag{10}$$

The plot of Eq.(10) w.r.t θ using Eq.(9b) is shown in Fig.(3) for different speeds of particle shows the change in $\frac{\mathrm{d}P}{\mathrm{d}\Omega}$ in synchrotron regime with right lobe blue shifted. Also the axial symmetry of the distribution about **a** in cyclotron radiation is broken in case of synchrotron radiation.

In non-relativistic speeds when the radiation even in the lab frame is in the cyclotron regime, the cone over which most of the power is radiated is $\theta = \pm\pi/2$. But in the synchrotron regime, only the longer lobe in the right(shown in Fig.(3)) emits most of the power. The angle of that cone is given by substituting $\theta = \pi/2$ in Eq.(8b):

$$\theta_{max} = \cos^{-1}\beta_v = \sin^{-1}\frac{1}{\gamma_v} \tag{11}$$

Thus an observer in the plane plane of motion observes the synchrotron radiation *more pulsed than uniform* compared to cyclotron radiation.

CONCLUSION

The effects on the various attributes of the radiation in cyclotron to synchrotron radiation could also be derived by working on lab frame directly from Lienard-Wiechert fields. But other than tedious algebra, the deeper phenomenological physics due to with the transition occurs with speed, is concealed by that conventional procedure. Working in a particle frame by Lorentz transforming the uniform magnetic field shows that the radiation pattern is no different than that of a point charge accelerating from rest. Lienerd-Wiechert fields gives a fairly intuitive angular distribution of power that is azimuthally symmetric and increases in direction where projection of acceleration is more. Periodicity and homogeneity gave rise to monochromaticity. The 4-vector $(\frac{\omega}{c}, \mathbf{k})$ when Lorentz transformed to lab frame shows blue and red shifting. Using the intensity-(no. density of photon)*ω^2 equivalence arguments, a power law for the lab frame was derived that showed the pulsed observation of synchrotron radiation to an observer which is evident from astrophysical observations.

Bibliography

[Cai12] Robert A Cairns. *Plasma physics*. Springer Science & Business Media, 2012.

[Zan13] Andrew Zangwill. *Modern electrodynamics*. Cambridge University Press, 2013.